LEFSCHETZ PROPERTIES AND THE VERONESE CONSTRUCTION

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Abstract. In this paper, we investigate Lefschetz properties of Veronese subalgebras. We show that, for a sufficiently large r, the r^{th} Veronese subalgebra of a Cohen-Macaulay standard graded K-algebra has properties similar to the weak and strong Lefschetz properties, which we call the 'almost weak' and 'almost strong' Lefschetz properties. By using this result, we obtain new results on h- and g-polynomials of Veronese subalgebras.

1. Introduction

Let K be a field of characteristic 0. For a standard graded (commutative) K-algebra $A = \bigoplus_{i\geq 0} A_i$ and for an integer $r \geq 1$, the K-algebra $A^{\langle r\rangle} := \bigoplus_{i\geq 0} A_{ir}$ is called the r^{th} *Veronese subalgebra* of A. Quite recently, it has been of interest to study h-polynomials of Veronese subalgebras [1, 2, 10]. In this paper, we investigate Lefschetz properties of Veronese subalgebras of Cohen-Macaulay standard graded K-algebras and obtain new results on h- and g-polynomials of Veronese subalgebras.

We first recall some basics on Hilbert series and h-polynomials. The *Hilbert series* of a standard graded K-algebra $A = \bigoplus_{i\geq 0} A_i$ is the formal power series $\mathrm{Hilb}(A,t) := \sum_{i\geq 0}(\dim_K A_i)t^i$. It is known that $\mathrm{Hilb}(A,t)$ is a rational function of the form $\mathrm{Hilb}(A,t) = (h_0 + h_1 t + \cdots + h_s t^s)/(1-t)^d$, where each h_i is an integer and where $d = \dim A$ is the Krull dimension of A (see e. g., [3, §4.1]). The polynomial

$$h_A(t) := h_0 + h_1 t + \cdots + h_s t^s$$

and the polynomial

$$g_A(t) := h_0 + (h_1 - h_0)t + \cdots + (h_{\lfloor \frac{s}{2}\rfloor} - h_{\lfloor \frac{s}{2}\rfloor - 1})t^{\lfloor \frac{s}{2}\rfloor}$$

are called the *h-polynomial* of A and the *g-polynomial* of A, respectively. Here, $\lfloor x\rfloor$ denotes the integer part of x.

For h-polynomials of Veronese subalgebras, Brenti and Welker [2, Corollary 1.6] proved that if $h_A(t) \in \mathbb{Z}_{\geq 0}[t]$, then, for sufficiently large r, the polynomial $h_{A^{\langle r\rangle}}(t)$ has only real zeros, and in particular, the coefficient sequence of $h_{A^{\langle r\rangle}}(t)$ is unimodal and log-concave. Moreover, it was proved in [10] that if $h_A(t) \in \mathbb{Z}_{\geq 0}[t]$ and if $r \geq \max\{\dim A, \deg h_A(t)\}$, then the g-polynomial of $A^{\langle r\rangle}$ is the f-polynomial of a simplicial complex. Algebraically, the unimodality of the h-polynomial of a graded K-algebra is closely related to Lefschetz properties of Artinian graded K-algebras. One of the main purposes of this paper is to find a connection between Lefschetz properties and the Veronese construction.

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We first consider the k-Lefschetz property and almost strong Lefschetz property, introduced in [9]. Let $A = \bigoplus_{i=0}^{s} A_i$ be a standard graded Artinian K-algebra, where $\dim_K A_s > 0$. For an integer $k \geq 1$, we say that A has the *k-Lefschetz property* if there is a linear form $w \in A_1$ such that the multiplication $w^{k-2i} : A_i \to A_{k-i} : \ p \mapsto w^{k-2i}p$ is injective for $0 \leq i \leq \lfloor \frac{k-1}{2} \rfloor$. The linear form w is referred to as a *k-Lefschetz element* for A. If A has the $(s-1)$-Lefschetz property, then we call it *almost strong Lefschetz*. An important consequence of the almost strong Lefschetz property is that if A is almost strong Lefschetz, then the multiplication $w : A_i \to A_{i+1}$ is injective for $0 \leq i \leq \lfloor \frac{s}{2} \rfloor - 1$ and the coefficient sequence of $g_A(t)$ becomes an M-sequence, namely, there is a standard graded Artinian K-algebra B such that $g_A(t) = \mathrm{Hilb}(B, t)$.

We use Lefschetz properties to study h-polynomials of $A^{\langle r \rangle}$ in the following way. Let A be a Cohen-Macaulay standard graded K-algebra of dimension d. A *linear system of parameters* (l.s.o.p. for short) for A is a sequence $\Theta = \theta_1, \dots, \theta_d \in A_1$ such that $\dim_K A/\Theta A < \infty$. Note that an l.s.o.p. for A exists if K is infinite, see e.g., [12]. For an l.s.o.p. $\Theta = \theta_1, \dots, \theta_d$ for A and for an integer $r \geq 1$, we write

$$A_{\Theta}^{\langle r \rangle} := A^{\langle r \rangle} / (\theta_1^r A^{\langle r \rangle} + \cdots + \theta_d^r A^{\langle r \rangle}).$$

We will see in Section 2, that $\theta_1^r, \dots, \theta_d^r$ is an l.s.o.p. for $A^{\langle r \rangle}$, and that the Hilbert series of $A_{\Theta}^{\langle r \rangle}$ is equal to the h-polynomial of $A^{\langle r \rangle}$. As a consequence, the h-polynomial of $A^{\langle r \rangle}$ can be analyzed via Lefschetz properties for $A_{\Theta}^{\langle r \rangle}$. Our first result is the following.

Theorem 1.1. *Let A be a Cohen-Macaulay standard graded K-algebra of dimension d and let $\Theta = \theta_1, \dots, \theta_d$ be an l.s.o.p. for A. Let $r \geq 1$ be an integer and $s = \lfloor \frac{(r-1)d}{r} \rfloor$. Then $A_{\Theta}^{\langle r \rangle}$ has the s-Lefschetz property. Moreover, if $r \geq \deg h_A(t)$, then $A_{\Theta}^{\langle r \rangle}$ is almost strong Lefschetz.*

In commutative algebra, the study of the weak Lefschetz property of Artinian graded K-algebras has shown to be of great interest. Recall, that a standard graded Artinian K-algebra $A = \bigoplus_{i=0}^{s} A_i$ is said to have the *weak Lefschetz property* if there is a linear form $w \in A_1$ such that the multiplication map $w : A_i \to A_{i+1}$ is either injective or surjective for all $i \geq 0$. Also, we say that A is *almost weak Lefschetz* if there is a $1 \leq p < s$ and a linear form $w \in A_1$ such that the multiplication map $w : A_i \to A_{i+1}$ is injective for $0 \leq i \leq p-1$ and is surjective for $i \geq p+1$. Note that we do not set any condition on the multiplication map $w : A_p \to A_{p+1}$. If the multiplication map $w : A_p \to A_{p+1}$ is neither injective nor surjective, then the integer p will be referred to as the *gap* of A (w.r.t. w). We obtain the following result for the almost weak and the weak Lefschetz property.

Theorem 1.2. *Let A be a Cohen-Macaulay standard graded K-algebra of dimension d and let $\Theta = \theta_1, \dots, \theta_d$ be an l.s.o.p. for A.*

(i) *If $r \geq \deg h_A(t)$, then $A_{\Theta}^{\langle r \rangle}$ is almost weak Lefschetz.*
(ii) *If d is even and $r \geq \max\{d, 2\deg h_A(t) - d\}$, then $A_{\Theta}^{\langle r \rangle}$ has the weak Lefschetz property.*

(iii) *If d is odd, $r \geq \frac{d}{2}$ and $\deg h_A(t) \leq \frac{d}{2}$, then $A_\Theta^{\langle r\rangle}$ has the weak Lefschetz property.*

In Section 2, for the almost weak Lefschetz property, we will show a result which is somewhat stronger, showing in particular that for $d \leq \deg h_A(t)$ a weaker assumption on r is sufficient for guaranteeing the almost weak Lefschetz property.

We say that a polynomial $h_0 + h_1 t + \cdots + h_s t^s \in \mathbb{Z}_{\geq 0}[t]$ is *unimodal* if there is a $1 \leq p \leq s$ such that $h_0 \leq h_1 \leq \cdots \leq h_p \geq h_{p+1} \geq \cdots \geq h_s$. Clearly, if a standard graded Artinian K-algebra A is almost weak Lefschetz, then the h-polynomial of A is unimodal. By using Theorem 1.1 and Theorem 1.2 together with a simple Gröbner basis argument, we also prove the following result on h-polynomials.

Theorem 1.3. *Let A be a Cohen-Macaulay standard graded K-algebra of dimension d. Let $r \geq 1$ be an integer and $s = \lfloor \frac{(r-1)d}{r} \rfloor$.*

(i) *If $r \geq \frac{1}{2}(\deg h_A(t) + 1)$, then $h_{A^{\langle r\rangle}}(t)$ is the f-polynomial of a flag simplicial complex.*
(ii) *If $r \geq \deg h_A(t)$, then $h_{A^{\langle r\rangle}}(t)$ is unimodal and $g_{A^{\langle r\rangle}}(t)$ is the f-polynomial of a simplicial complex.*
(iii) *Let $h_{A^{\langle r\rangle}}(t) = \sum_{i\geq 0} h_i^{\langle r\rangle} t^i$. Then $h_i^{\langle r\rangle} \leq h_{s-i}^{\langle r\rangle}$ for all $i \leq \frac{s}{2}$.*

2. Lefschetz Properties

In this section, we study Lefschetz properties of $A_\Theta^{\langle r\rangle}$. In particular, we will provide the proofs of Theorem 1.1 and Theorem 1.2.

We start to fix some notation, which we will use throughout this section. In the following, we consider a Cohen-Macaulay standard graded K-algebra A of dimension d together with an l.s.o.p. $\Theta = \theta_1, \ldots, \theta_d$ for A. To prove Theorem 1.1 and Theorem 1.2, we use the following observation, which relates the Hilbert series of $A_\Theta^{\langle r\rangle}$ to the h-polynomial of the r^{th} Veronese subalgebra of A: By Cohen-Macaulayness of A, Θ is not only an l.s.o.p. but also a regular sequence for A. Hence, A is a finitely generated and free $K[\theta_1, \ldots, \theta_d]$-module. In particular, there exist homogeneous elements $u_1, \ldots, u_m$ of A such that we have the decomposition

$$A = \bigoplus_{j=1}^{m} u_j \cdot K[\theta_1, \ldots, \theta_d] \tag{1}$$

as $K[\theta_1, \ldots, \theta_d]$-modules. Note that $u_1, \ldots, u_m$ is a K-basis of $A/\Theta A$ (see e.g., [12, Chapter 1]). Moreover, since the Hilbert series of $A/\Theta A$ is equal to the h-polynomial of A (cf., [3, Remark 4.1.11]), we have

$$\deg u_j \leq \deg h_A(t)$$

for all $1 \leq j \leq m$. Let $r \geq 1$ be an integer. We will show that $\theta_1^r, \ldots, \theta_d^r$ is an l.s.o.p. for $A^{\langle r\rangle}$. From (1) we infer that the r^{th} Veronese subalgebra $A^{\langle r\rangle}$ decomposes as

$$A^{\langle r\rangle} = \bigoplus_{j=1}^{m} u_j \cdot \left(\bigoplus_{i\geq 0} K[\theta_1, \ldots, \theta_d]_{ir-\deg u_j} \right),$$

where we set $K[\theta_1,\dots,\theta_d]_k := \{0\}$ if $k<0$. And for the quotient $A^{\langle r\rangle}_\Theta = A^{\langle r\rangle}/(\theta_1^r A^{\langle r\rangle} + \cdots + \theta_d^r A^{\langle r\rangle})$ we obtain

$$A^{\langle r\rangle}_\Theta = \bigoplus_{j=1}^m u_j \cdot \left(\bigoplus_{i\geq 0} \big(K[\theta_1,\dots,\theta_d]/(\theta_1^r,\dots,\theta_d^r)\big)_{ir-\deg u_j} \right) \tag{2}$$

as $K[\theta_1,\dots,\theta_d]^{\langle r\rangle}$-modules. Being the grading of $A^{\langle r\rangle}_\Theta$ induced by the usual $\mathbb{Z}$-grading of $K[\theta_1,\dots,\theta_d]$, we know that the homogeneous component of $A^{\langle r\rangle}_\Theta$ of degree i is given by

$$\bigoplus_{j=1}^m u_j \cdot \big(K[\theta_1,\dots,\theta_d]/(\theta_1^r,\dots,\theta_d^r)\big)_{ir-\deg u_j}. \tag{3}$$

Since the right-hand side of (2) has finite length, we conclude that $\theta_1^r,\dots,\theta_d^r$ is an l.s.o.p. for $A^{\langle r\rangle}$. This together with the fact that the Cohen-Macaulay property is preserved under taking Veronese subalgebras (cf. [7, Chapter 3]) implies that the Hilbert series of $A^{\langle r\rangle}_\Theta$ equals the h-polynomial of $A^{\langle r\rangle}$. In particular, since $\max\{\deg u_j \mid 1\leq j\leq m\} = \deg h_A(t)$ and since the maximum degree in $K[\theta_1,\dots,\theta_d]/(\theta_1^r,\dots,\theta_d^r)$ is $(r-1)d$, it follows directly from (3) that

$$\deg h_{A^{\langle r\rangle}}(t) = \left\lfloor \frac{d(r-1)+\deg h_A(t)}{r} \right\rfloor. \tag{4}$$

Note that the above equation (4) holds for any standard graded K-algebra A whose h-polynomial has non-negative coefficients (e.g., use [2, Theorem 1.2]).

For the proof of Theorem 1.1 and Theorem 1.2, we need the following fact proved by Stanley [11] and Watanabe [13].

Lemma 2.1. *Let K be a field of characteristic 0 and let $r\geq 1$ be an integer. For integers $0\leq i<j$, the multiplication map*

$$\begin{aligned} \times(x_1+\cdots+x_d)^{j-i} : (K[x_1,\dots,x_d]/(x_1^r,\dots,x_d^r))_i \quad &\to \quad (K[x_1,\dots,x_d]/(x_1^r,\dots,x_d^r))_j \\ p \quad &\mapsto \quad (x_1+\cdots+x_d)^{j-i}\cdot p \end{aligned}$$

is injective if $i+j\leq (r-1)d$ and is surjective if $i+j\geq (r-1)d$.

We have now laid the necessary foundations for giving the proof of Theorem 1.1.

Proof of Theorem 1.1: Let $w=(\theta_1+\cdots+\theta_d)^r$. We prove that w is an s-Lefschetz element of $A^{\langle r\rangle}_\Theta$, namely, we will show that the multiplication

$$\times w^{s-2i} : (A^{\langle r\rangle}_\Theta)_i \to (A^{\langle r\rangle}_\Theta)_{s-i}$$

is injective for $0\leq i\leq \lfloor \frac{s-1}{2}\rfloor$. By the decomposition (2), it is enough to prove that, for $1\leq j\leq m$, the multiplication

$$\times w^{s-2i} : \big(K[\theta_1,\dots,\theta_d]/(\theta_1^r,\dots,\theta_d^r)\big)_{ir-\deg u_j} \to \big(K[\theta_1,\dots,\theta_d]/(\theta_1^r,\dots,\theta_d^r)\big)_{(s-i)r-\deg u_j}$$

is injective for the same i. The desired injectivity follows from Lemma 2.1 since $ir-\deg u_j+(s-i)r-\deg u_j\leq sr\leq (r-1)d$.

Finally, if $r \geq \deg h_A(t)$, then $\deg \mathrm{Hilb}(A_\Theta^{\langle r\rangle}, t) = \deg h_{A^{\langle r\rangle}}(t) \leq s+1$ by (4), which implies that $A_\Theta^{\langle r\rangle}$ is almost strong Lefschetz. $\square$

We now proceed to the proof of Theorem 1.2. Part (i), i. e., the statement concerning the almost weak Lefschetz property, follows from the following stronger result.

Theorem 2.2. *Let A be a Cohen-Macaulay standard graded K-algebra of dimension d and let $\Theta = \theta_1, \dots, \theta_d$ be an l.s.o.p. for A. Then $A_\Theta^{\langle r\rangle}$ is almost weak Lefschetz if*

(a) *d is even and one of the following conditions holds:*
  (i) *$d \leq \frac{1}{2}\deg h_A(t)$ and $r \geq \frac{2\deg h_A(t) - d}{3}$,*
  (ii) *$\frac{1}{2}\deg h_A(t) \leq d \leq \deg h_A(t)$ and $r \geq d$,*
  (iii) *$\deg h_A(t) \leq d \leq \frac{3}{2}\deg h_A(t)$ and $r \geq 2\deg h_A(t) - d$,*
  (iv) *$\frac{3}{2}\deg h_A(t) \leq d \leq 3\deg h_A(t)$ and $r \geq \frac{d}{3}$,*
  (v) *$d \geq 3\deg h_A(t)$ and $r \geq \deg h_A(t)$, or,*

(b) *d is odd and one of the following conditions holds:*
  (i) *$d \leq \deg h_A(t)$ and $r \geq \deg h_A(t) - \frac{d}{2}$,*
  (ii) *$\deg h_A(t) \leq d \leq 2\deg h_A(t)$ and $r \geq \frac{d}{2}$,*
  (iii) *$d \geq 2\deg h_A(t)$ and $r \geq \deg h_A(t)$.*

Proof. Before providing the proofs for each set of conditions separately, we start with a general discussion that can be used in all cases. Let $w := (\theta_1 + \cdots + \theta_d)^r$. Our aim is to show that in all parts of the theorem, w can be used as an almost weak Lefschetz element for $A_\Theta^{\langle r\rangle}$. Using the same notations as at the beginning of this section, we know from (2) that as $K[\theta_1, \dots, \theta_d]^{\langle r\rangle}$-modules, we have the decomposition

$$A_\Theta^{\langle r\rangle} = \bigoplus_{j=1}^{m} u_j \left(\bigoplus_{i\geq 0} \left(K[\theta_1, \dots, \theta_d]/(\theta_1^r, \dots, \theta_d^r) \right)_{ir - \deg u_j} \right).$$

Thus, in order to show that the multiplication

$$\times w : (A_\Theta^{\langle r\rangle})_i \to (A_\Theta^{\langle r\rangle})_{i+1}$$

is injective and surjective for a certain $i \geq 0$, it suffices to show that for all $1 \leq j \leq m$ the multiplication

$$\times w : \left(K[\theta_1, \dots, \theta_d]/(\theta_1^r, \dots, \theta_d^r) \right)_{ir - \deg u_j} \to \left(K[\theta_1, \dots, \theta_d]/(\theta_1^r, \dots, \theta_d^r) \right)_{(i+1)r - \deg u_j} \tag{5}$$

is injective and surjective, respectively for the same i.

We first consider case (a) (i). Suppose that d is even, $d \leq \frac{1}{2}\deg h_A(t)$ and $r \geq \frac{2\deg h_A(t) - d}{3}$. Combining the latter two conditions in particular yields $r \geq d$. Our aim is to use Lemma 2.1. We first show that the multiplication in (5) is injective for $0 \leq i \leq \frac{d}{2} - 1$ and for all $1 \leq j \leq m$. For all $1 \leq j \leq m$ it holds that

$$2ir + r - 2\deg u_j \leq dr - r \leq (r-1)d + d - r \leq (r-1)d,$$

where the first and the last inequality follow from $\deg u_j \geq 0$ for $1 \leq j \leq m$ and $r \geq d$, respectively. Hence, Lemma 2.1 implies the desired injectivity.

Next, we show that the multiplication in (5) is surjective for $i \geq \frac{d}{2}+1$. As in the previous case, for $1 \leq j \leq m$ we compute

$$2ir + r - 2\deg u_j \geq dr + 3r - 2\deg h_A(t) \geq (r-1)d,$$

where for the first inequality we use that $\deg u_j \leq \deg h_A(t)$ for $1 \leq j \leq m$, and the last inequality holds since $r \geq \frac{2\deg h_A(t)-d}{3}$. Surjectivity now follows from Lemma 2.1.

The cases (a) (ii) – (iv) and (b) (i) – (ii) follow from almost literally the same arguments, taking into account the different ranges and bounds for d and r, respectively, as well as the different location of the gap. Indeed, if there is a gap, then it is at position $\frac{d}{2}$ in the cases (a) (i) – (ii), and at position $\frac{d}{2}-1$ in the cases (iii) – (iv). In the situation of (b) (i) – (ii), the gap – if existing – lies at position $\frac{d-1}{2}$.

The cases (a) (v) and (b) (iii) have to be treated slightly differently. Let $s = \lfloor \frac{(r-1)d}{r} \rfloor$. By an analog reasoning as for the other cases one infers that the multiplication in (5) is surjective for $i \geq \frac{s}{2}+1$. On the other hand, Theorem 1.1 says that $A_\Theta^{\langle r \rangle}$ is s-Lefschetz. In particular the multiplication map in (5) is injective for $i \leq \frac{s-1}{2}$. Hence, we conclude that $A_\Theta^{\langle r \rangle}$ is almost weak Lefschetz with a possible gap at position $\lfloor \frac{s+1}{2} \rfloor$. $\square$

We want to remark that the arguments in the above proof do only depend on the effective size of r and not on the precise relation between d and $\deg h_A(t)$. Moreover, the proofs of (a) (iv) and (b) (iii) do not use the fact that $d \leq 3\deg h_A(t)$ and $d \leq 2\deg h_A(t)$, respectively. We only include these restrictions since for $d > \deg h_A(t)$ part (a) (v) and part (b) (iii) provide better, i. e., smaller bounds for r. In particular, this allows us to conclude, that if d is even, the gap – if existing – is at position $\frac{d}{2}$ if $r \geq \max\{d, \frac{2\deg h_A(t)-d}{3}\}$ and at position $\frac{d}{2}+1$ if $r \geq \max\{\frac{d}{3}, 2\deg h_A(t)-d\}$. If d is odd and $r \geq \max\{\frac{d}{2}, \deg h_A(t) - \frac{d}{2}\}$, the gap is at position $\frac{d-1}{2}$. This will be relevant for the proof of Theorem 1.2 (ii) and (iii).

Proof of Theorem 1.2: Part (i) can readily be deduced from Theorem 2.2. To show part (ii), note that – independent of d – it follows from Theorem 2.2 (a) (i) –(iv) and the discussion preceding this proof that $A_\Theta^{\langle r \rangle}$ is almost weak Lefschetz. Since there can exist at most one gap, we infer from the mentioned discussion that $A_\Theta^{\langle r \rangle}$ is indeed weak Lefschetz.

For part (iii), let $r \geq \max\{\frac{d}{2}, \deg h_A(t) - \frac{d}{2}\}$. By Theorem 2.2 and the discussion preceding this proof, we know that $A_\Theta^{\langle r \rangle}$ is almost weak Lefschetz with a possible gap being at position $\frac{d-1}{2}$. Assume, in addition, that $\deg h_A(t) \leq \frac{d}{2}$. We claim that the multiplication map

$$\times w : (A_\Theta^{\langle r \rangle})_{\frac{d-1}{2}} \to (A_\Theta^{\langle r \rangle})_{\frac{d-1}{2}+1}$$

is surjective. The desired surjectivity follows from Lemma 2.1 since for all $1 \leq j \leq m$ it holds that

$$r\left(\frac{d-1}{2} + \frac{d-1}{2} + 1\right) - 2\deg u_j \geq rd - 2\deg h_A(t) \geq (r-1)d.$$

We conclude that $A_\Theta^{\langle r \rangle}$ has the weak Lefschetz property. $\square$

Remark 2.3. Theorem 1.2(ii) says that, for any even dimensional Cohen-Macaulay graded K-algebra A, the algebra $A_\Theta^{\langle r\rangle}$ has the weak Lefschetz property for $r \gg 0$. Unfortunately, this fact does not hold for odd dimensional Cohen-Macaulay graded K-algebras. Let $A = K[x_1,\dots,x_8]/((x_1^2,x_1x_2,x_1x_3,x_1x_4,x_1x_5)+(x_2,x_3,x_4,x_5)^3)$. Then A is a Cohen-Macaulay graded K-algebra of dimension 3 with the h-polynomial $h_A(t) = 1+5t+10t^2$ and $\Theta = x_6,x_7,x_8$ is an l.s.o.p. for A, but $A_\Theta^{\langle r\rangle}$ does not have the weak Lefschetz property for any $r \geq 3$.

If $r \geq 3$, then the h-polynomial $h_{A^{\langle r\rangle}}(t) = h_0^{\langle r\rangle} + h_1^{\langle r\rangle}t + h_2^{\langle r\rangle}t^2$ of $A^{\langle r\rangle}$ satisfies $h_0^{\langle r\rangle} < h_1^{\langle r\rangle} < h_2^{\langle r\rangle}$. However, there are no linear forms w such that $\times w : (A_\Theta^{\langle r\rangle})_1 \to (A_\Theta^{\langle r\rangle})_2$ is injective. Consider the K-vector spaces $V = x_1(K[x_6,x_7,x_8]/(x_6^r,x_7^r,x_8^r))_{r-1} \subset (A_\Theta^{\langle r\rangle})_1$ and $W = x_1(K[x_6,x_7,x_8]/(x_6^r,x_7^r,x_8^r))_{2r-1} \subset (A_\Theta^{\langle r\rangle})_2$. Then, for any linear form $w \in A_\Theta^{\langle r\rangle}$ we have $wV \subset W$, since $x_1x_i = 0$ in A for $i = 1,2,\dots,5$, but

$$\begin{aligned} \dim_K V &= \dim_K(K[x_6,x_7,x_8]/(x_6^r,x_7^r,x_8^r))_{r-1} \\ &> \dim_K(K[x_6,x_7,x_8]/(x_6^r,x_7^r,x_8^r))_{2r-1} = \dim_K W, \end{aligned}$$

where the inequality follows since $\dim_K(K[x_6,x_7,x_8]/(x_6^r,x_7^r,x_8^r))_{r-1} = \binom{r+1}{r-1}$ and $\dim_K(K[x_6,x_7,x_8]/(x_6^r,x_7^r,x_8^r))_{2r-1} = \dim_K(K[x_6,x_7,x_8]/(x_6^r,x_7^r,x_8^r))_{r-2} = \binom{r}{r-2}$. This fact implies that the multiplication $\times w : V \to W$ is not injective.

3. Consequences on h-vectors

In this section, we prove Theorem 1.3. Throughout this section, we let $S = K[x_1,\dots,x_n]$ be a standard graded polynomial ring over a field K. For an integer $r \geq 1$, let

$$T_{\langle r\rangle} = K[z_m : m \text{ is a monomial in } S \text{ of degree } r],$$

where each z_m is a variable. Then there is a natural ring homomorphism

$$\begin{array}{cccc} \phi_r: & T_{\langle r\rangle} & \longrightarrow & S^{\langle r\rangle} \\ & z_m & \mapsto & m. \end{array}$$

For a homogeneous ideal $I \subset S$, let $I^{\langle r\rangle} := \bigoplus_{j\geq 0} I_{jr}$. Then $I^{\langle r\rangle}$ is a graded ideal of $S^{\langle r\rangle}$ and $(S/I)^{\langle r\rangle} = S^{\langle r\rangle}/I^{\langle r\rangle}$. Also, the ring $(S/I)^{\langle r\rangle}$ is isomorphic to $T_{\langle r\rangle}/\phi_r^{-1}(I^{\langle r\rangle})$.

To prove the main result, we need a few known results on Gröbner bases of $\phi^{-1}(I^{\langle r\rangle})$ proved by Eisenbud, Reeves and Totaro [5]. We refer the readers to [4] for the basics on Gröbner basis theory.

Let $>_{\text{rev}}$ be the reverse lexicographic order on S induced by $x_1 > \cdots > x_n$, and let $\succ_{\text{rev}}$ be the reverse lexicographic order on $T_{\langle r\rangle}$ such that the ordering of the variables is defined by $z_m \succ_{\text{rev}} z_{m'}$ if $m >_{\text{rev}} m'$. For a monomial $m \in S$, we write $\max(m)$ (resp. $\min(m)$) for the largest (resp. smallest) integer i such that x_i divides m. We say that a monomial

$$u = z_{m_1}z_{m_2}\cdots z_{m_k} \in T_{\langle r\rangle},$$

where $m_1 >_{\text{rev}} \cdots >_{\text{rev}} m_k$, is *standard* if $\max(m_i) \leq \min(m_{i+1})$ for $1 \leq i \leq k-1$. The following fact was shown in the proof of [5, Proposition 6].

Lemma 3.1. *A monomial $u \in T_{\langle r\rangle}$ is standard if and only if $u \notin \text{in}_{\succ_{\text{rev}}}(\ker\phi_r)$.*

The above lemma implies the next result.

Lemma 3.2. *Let $r \geq 1$ and $1 \leq \ell \leq n$ be integers. Let $I \subset S$ be a monomial ideal and $J = I + (x_n^r, x_{n-1}^r, \dots, x_\ell^r)$. Then*

$$\mathrm{in}_{\succ_{\mathrm{rev}}} \phi_r^{-1}(J^{\langle r \rangle}) = \mathrm{in}_{\succ_{\mathrm{rev}}} \phi_r^{-1}(I^{\langle r \rangle}) + (z_{x_n^r}, \dots, z_{x_\ell^r}) + (z_m z'_m : mm' \in (x_n^r, \dots, x_\ell^r)).$$

Proof. It is clear that the left-hand side contains the right-hand side. We show that also the reverse inclusion holds. Let

$$u = z_{m_1} z_{m_2} \cdots z_{m_k} \in \mathrm{in}_{\succ_{\mathrm{rev}}} \phi_r^{-1}(J^{\langle r \rangle}),$$

be a monomial with $m_1 >_{\mathrm{rev}} \cdots >_{\mathrm{rev}} m_k$. We show that if $u \notin \mathrm{in}_{\succ_{\mathrm{rev}}} \phi_r^{-1}(I^{\langle r \rangle})$, then $u \in (z_{x_n^r}, \dots, z_{x_\ell^r}) + (z_m z_{m'} : mm' \in (x_n^r, \dots, x_\ell^r))$.

Since $u \notin \mathrm{in}_{\succ_{\mathrm{rev}}} \phi_r^{-1}(I^{\langle r \rangle})$, we have $u \notin \mathrm{in}_{\succ_{\mathrm{rev}}} \ker(\phi_r)$. Thus u is standard by Lemma 3.1. We claim $\phi_r(u) \in J^{\langle r \rangle}$. Let $f = u + v_1 + \cdots + v_m + g \in \phi_r^{-1}(J^{\langle r \rangle})$ be such that $\mathrm{in}_{\succ_{\mathrm{rev}}}(f) = u$, $g \in \ker(\phi_r)$ and $u, v_1, \dots, v_m$ are distinct standard monomials. Then $\phi_r(f) = \phi_r(u) + \phi_r(v_1) + \cdots + \phi_r(v_m) \in J^{\langle r \rangle}$. Since J is a monomial ideal and $\phi_r(u), \phi_r(v_1), \dots, \phi_r(v_m)$ are distinct monomials, we have $\phi_r(u) \in J^{\langle r \rangle}$.

Since, by assumption, $u \notin \phi_r^{-1}(I^{\langle r \rangle})$, we have

$$\phi_r(u) = m_1 m_2 \cdots m_k \in (x_n^r, \dots, x_\ell^r).$$

Thus, there is an $\ell \leq i \leq n$ such that x_i^r divides $\phi_r(u)$. If $\deg u = 1$, then u must be equal to $z_{x_i^r}$. If $\deg u > 1$, then, by the definition of a standard monomial, there is a $1 \leq j \leq k-1$ such that x_i^r divides $m_j m_{j+1}$. This proves the desired statement. □

A monomial ideal $I \subset S$ is called *stable* if, for any monomial $m \in I$, one has $m(x_i / x_{\max(m)}) \in I$ for any $i < \max(m)$. The following facts are known.

Lemma 3.3.

(i) *For any Cohen-Macaulay standard graded K-algebra A with $\dim_K A_1 \leq n$, there is a stable monomial ideal $J \subset S$ such that S/J is Cohen-Macaulay and S/J has the same Hilbert series as A.*

(ii) *Let $I \subset S$ be a stable monomial ideal such that S/I is a Cohen-Macaulay graded K-algebra of dimension d. Then $x_n, x_{n-1}, \dots, x_{n-d+1}$ is a linear system of parameters for S/I and I is generated by monomials of degree $\leq \deg h_{S/I}(t) + 1$.*

Proof. We only sketch the proof since the statements are well-known in commutative algebra. For any standard graded K-algebra A with $\dim_K A_1 \leq n$, there is a homogeneous ideal $I \subset S$ such that S/I is isomorphic to A as graded K-algebra. Then (i) follows from [8, Theorem 2].

Suppose that I is a stable monomial ideal such that S/I is Cohen-Macaulay. A result of Eliahou and Kervaire [6] shows that I is generated by monomials in $K[x_1, \dots, x_{n-d}]$. This shows that $x_n, x_{n-1}, \dots, x_{n-d+1}$ is a regular sequence of S/I and, therefore, is an l.s.o.p. for S/I. Also, since the h-polynomial of S/I is equal to the Hilbert series of $S/(I + (x_n, x_{n-1}, \dots, x_{n-d+1}))$, I contains all monomials in $K[x_1, \dots, x_{n-d}]$ of degree $\deg h_{S/I}(t) + 1$. Since I is generated by monomials in $K[x_1, \dots, x_{n-d}]$, I is generated by monomials of degree $\leq \deg h_{S/I}(t) + 1$. □

For the proof of Theorem 1.3 we will use the following result for Veronese algebras of the quotient of a stable monomial ideal, which was proven by Eisenbud, Reeves and Totaro [5, Theorem 8].

Lemma 3.4 (Eisenbud-Reeves-Totaro). *Let $I \subset S$ be a stable monomial ideal generated by monomials of degree $\leq s$. If $r \geq \frac{s}{2}$, then $\mathrm{in}_{\succ_{\mathrm{rev}}} \phi_r^{-1}(I^{\langle r \rangle})$ is generated by monomials of degree ≤ 2.*

Now we are in the position to prove Theorem 1.3. Recall that a *simplicial complex* Δ on $[n] := \{1, 2, \dots, n\}$ is a collection of subsets of $[n]$ satisfying that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. A simplicial complex is said to be *flag* if every minimal non-face of Δ has cardinality ≤ 2. For a simplicial complex Δ, we write $f_i(\Delta)$ for the number of elements $F \in \Delta$ with $|F| = i + 1$. The *f-polynomial* of Δ is the polynomial $f(\Delta, t) = \sum_{i \geq 0} f_{i-1}(\Delta) t^i$, where $f_{-1}(\Delta) := 1$. The f-polynomial of Δ can be expressed in an algebraic way. Indeed, the f-polynomial of a simplicial complex Δ on $[n]$ is equal to the Hilbert series of $S/((x_F : F \notin \Delta) + (x_1^2, \dots, x_n^2))$, where $x_F := \prod_{i \in F} x_i$. Moreover, Δ is flag if and only if the ideal $(x_F : F \notin \Delta) + (x_1^2, \dots, x_n^2)$ is generated by monomials of degree ≤ 2.

Proof of Theorem 1.3: Part (iii) immediately follows from Theorem 1.1. The unimodality of (ii) is a direct consequence of Theorem 1.2. We prove (i) and the remaining part of (ii).

Fix $r \geq 1$. Since the Hilbert series of $A^{\langle r \rangle}$ only depends on r and the Hilbert series of A, by Lemma 3.3 (i), we may assume that $A = S/I$, where I is a stable monomial ideal. Let $\Theta = x_n, x_{n-1}, \dots, x_{n-d+1}$ and $J = I + (x_n^r, \dots, x_{n-d+1}^r)$. Then, by Lemma 3.3 (ii), Θ is an l.s.o.p. for $A = S/I$ and

$$A_\Theta^{\langle r \rangle} = S^{\langle r \rangle} / J^{\langle r \rangle} \cong T_{\langle r \rangle} / \phi_r^{-1}(J^{\langle r \rangle}).$$

We now prove (i). Suppose $r \geq \frac{1}{2}(\deg h_A(t) + 1)$. Let Δ be the set of monomials in $T_{\langle r \rangle}$, which are not contained in $\mathrm{in}_{\succ_{\mathrm{rev}}}(\phi_r^{-1}(J^{\langle r \rangle}))$. By Lemma 3.3 (ii), I is generated by monomials of degree $\leq \deg h_A(t) + 1$. Then Lemma 3.2 and Lemma 3.4 say that $\mathrm{in}_{\succ_{\mathrm{rev}}}(\phi_r^{-1}(J^{\langle r \rangle}))$ is generated by monomials of degree ≤ 2. This fact shows that $\mathrm{in}_{\succ_{\mathrm{rev}}}(\phi_r^{-1}(J^{\langle r \rangle}))$ contains z_m^2 for any variable z_m of $T_{\langle r \rangle}$ since $T_{\langle r \rangle} / \phi_r^{-1}(J^{\langle r \rangle})$ is Artinian. This implies that Δ is a set of squarefree monomials. Thus, we may regard Δ as a simplicial complex. Moreover, since

$$\mathrm{in}_{\succ_{\mathrm{rev}}}(\phi_r^{-1}(J^{\langle r \rangle})) = (u : u \text{ is a monomial in } T_{\langle r \rangle} \text{ with } u \notin \Delta)$$

is generated by monomials of degree ≤ 2, Δ is a flag simplicial complex. Also, by the construction of Δ, we have

$$f(\Delta, t) = \mathrm{Hilb}(T_{\langle r \rangle} / \phi_r^{-1}(J^{\langle r \rangle}), t) = \mathrm{Hilb}(A_\Theta^{\langle r \rangle}, t) = h_{A^{\langle r \rangle}}(t),$$

which proves (i).

Finally, we prove the second part of (ii). Suppose $r \geq \deg h_A(t)$. Let $\lambda = \deg h_{A^{\langle r \rangle}}(t)$. By Theorem 1.2 and the proof of Theorem 2.2, there is a linear form

$w \in (S^{\langle r \rangle})_1 = S_r$ such that

$$g_{A^{\langle r \rangle}}(t) = \sum_{i=0}^{\lfloor \frac{\lambda}{2} \rfloor} \big(\dim_K (A_\Theta^{\langle r \rangle}/wA_\Theta^{\langle r \rangle})_i \big) t^i. \tag{6}$$

Observe

$$A_\Theta^{\langle r \rangle}/wA_\Theta^{\langle r \rangle} \cong T_{\langle r \rangle}/\phi_r^{-1}(J^{\langle r \rangle} + (w)^{\langle r \rangle}). \tag{7}$$

Let Γ be the set of monomials of degree $\leq \lfloor \frac{\lambda}{2} \rfloor$ which are not in $\mathrm{in}_{\succ_{\mathrm{rev}}} \phi_r^{-1}(J^{\langle r \rangle} + (w)^{\langle r \rangle})$. As we have already seen in the proof of (i), $\mathrm{in}_{\succ_{\mathrm{rev}}} \phi_r^{-1}(J^{\langle r \rangle})$ contains z_m^2 for any variable z_m of $T_{\langle r \rangle}$. Thus Γ can be regarded as a simplicial complex. Then, (6) and (7) say that $g_{A^{\langle r \rangle}}(t)$ is equal to the f-polynomial of Γ, as desired. $\square$

References

[1] M. Beck and A. Stapledon. On the log-concavity of Hilbert series of Veronese subrings and Ehrhart series. *Math. Z.*, 264:195–2007, 2010.

[2] F. Brenti and V. Welker. The Veronese construction for formal power series and graded algebras. *Adv. Appl. Math.*, 42:545–556, 2009.

[3] W. Bruns and J. Herzog. *Cohen-Macaulay rings. Rev. ed.*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1998.

[4] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer-Verlag, 1997.

[5] D. Eisenbud, A. Reeves, and B. Totaro. Initial ideals, Veronese subrings, and rates of algebras. *Adv. Math.*, 109(2):168–187, 1994.

[6] S. Eliahou and M. Kervaire. Minimal resolutions of some monomial ideals. *J. Algebra*, 129:1–25, 1990.

[7] S. Goto and K. Watanabe. On graded rings. i. *J. Math. Soc. Japan*, 30:179–213, 1978.

[8] S. Iyenger and K. Pardue. Maximal minimal resolutions. *J. reine angew. Math.*, 512:27–46, 1999.

[9] M. Kubitzke and E. Nevo. The Lefschetz property for barycentric subdivisions of shellable complexes. *Trans. Am. Math. Soc.*, 361(11):6151–6163, 2009.

[10] M. Kubitzke and V. Welker. Enumerative g-theorems for the Veronese construction for formal power series and graded algebras. *http://arxiv.org/abs/1108.2852*, 2011.

[11] R.P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods*, 1:168–184, 1980.

[12] R.P. Stanley. *Combinatorics and Commutative Algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[13] J. Watanabe. The Dilworth number of Artinian rings and finite posets with rank function. In *Commutative algebra and combinatorics (Kyoto, 1985)*, volume 11 of *Adv. Stud. Pure Math.*, pages 303–312. North-Holland, Amsterdam, 1987.

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